

NONLINEAR SECOND ORDER OSCILLATORS OFF RESONANCE AT CERTAIN FUNCTIONAL SPACES

ADOLFO ARROYO RABASA

ABSTRACT. We will deal with the existence of odd and T -periodic solutions of the scalar equation

$$(1) \quad u'' + g(u) = k(t),$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is an odd function and k is an odd and T -periodic function of mean zero. By putting (1) in means of

$$Lu = Nu,$$

where $Lu = u''$ is the linear part and $Nu = k - g(u)$ is the nonlinear part, generally if one denotes by P_T the continuous T -periodic functions and by Q_T the continuous T -periodic functions of mean zero, then

$$L : P_T \cap C^2 \rightarrow P_T$$

and one have that $\text{Ker}(L) = \mathbb{R}$ and $\text{Rank}(L) = Q_T$, which is clearly a resonant problem. We will consider the space of odd and T -periodic functions where one can avoid resonance. In this space we state two results of existence, one including a priori bounds and one of uniqueness. This results generalize the results obtained by Hamel in [2] on the periodic problem for the forced pendulum equation

1. OBSERVATIONS AT RESONANCE

We denote \mathcal{H} the set of continuous, odd and T -periodic functions.

$$\mathcal{H} := \{u \in P_T \mid u(t) = -u(-t) \ \forall t \in [0, T]\}.$$

Observation 1. \mathcal{H} is a complete normed space.

Lemma 1. *Let $u \in P_T$ be an odd function, then $\bar{u} := \frac{1}{T} \int_0^T u \, dt = 0$*

Proof. Let

$$\begin{aligned} \Omega^+ &:= \{t \in [-T/2, T/2] : u(t) > 0\}, \\ \Omega^- &:= \{t \in [-T/2, T/2] : u(t) < 0\}. \end{aligned}$$

If $t \in \Omega^+ \Rightarrow u(t) > 0 \Rightarrow -u(-t) > 0 \Rightarrow u(-t) < 0 \Rightarrow -t \in \Omega^-$, therefore $-\Omega^+ \subseteq \Omega^-$. In a similar way one proves that $-\Omega^- \subseteq \Omega^+$ and consequently

$$(2) \quad \Omega^+ = -\Omega^-.$$

From (2), it follows that

$$(3) \quad \bar{u} = \frac{1}{T} \left(\int_{\Omega^+} u(t) dt + \int_{\Omega^-} u(t) dt \right) = \frac{1}{T} \left(\int_{\Omega^-} u(-t) dt + \int_{\Omega^-} u(t) dt \right).$$

Finally, the odd property of u in (3) implies

$$\bar{u} = \frac{1}{T} \left(- \int_{\Omega^-} u(t) dt + \int_{\Omega^-} u(t) dt \right) = 0.$$

□

Observation 2. $\mathcal{H} \subset Q_T$.

Lemma 2. $L|_{\mathcal{H}} : \text{dom}(L) \cap \mathcal{H} \rightarrow \mathcal{H}$ is invertible.

From the above observation it is clear that $L(\mathcal{H}) \subset P_T$, therefore is necessary to prove that Lu is an odd function for all odd functions u .

Proof. For u' , we have

$$(4) \quad u'(t) = \lim_{h \rightarrow 0} \frac{u(t) - u(t+h)}{h} = \lim_{h \rightarrow 0} \frac{u(-t) - u(-t-h)}{-h} = u'(-t),$$

it follows that u' is an even function. Using (4) in u'' it follows that

$$u''(t) = \lim_{h \rightarrow 0} \frac{u'(t) - u'(t+h)}{h} = - \lim_{h \rightarrow 0} \frac{u'(-t) - u'(-t-h)}{-h} = -u''(-t).$$

This proves that $L|_{\mathcal{H}} : \text{dom}(L) \cap \mathcal{H} \rightarrow \mathcal{H}$ is well defined. It is well known from [3] the existence of an integral operator $S : Q_T \rightarrow P_T$, a right inverse of L such that

$$\|S(f)\|_{\infty} \leq \frac{T^2}{2} \|f\|_{\infty}$$

and $S(\mathcal{H}) \subseteq \text{dom}(L) \cap \mathcal{H}$, for it is sufficient to observe that $\text{Ker}(L|_{\mathcal{H}}) = \{0\}$. □

Lemma 2 tells us that (1) is a non-resonant problem in \mathcal{H} . Naturally, the question arises to answer when does $Nu \in \mathcal{H}$, in order to express any solution of (1) as a classic fixed point problem of the form

$$u = L^{-1}Nu.$$

From now on, let us put $L := L|_{\mathcal{H}}$.

2. EXISTENCE OF SOLUTIONS IN \mathcal{H} IN THE SUBLINEAR CASE

Since odd functions form are closed under composition we get

$$\overline{g(u)} = 0,$$

and even more

$$Nu = k - g(u) \in \mathcal{H}$$

if g is an odd function. The latter discussion lead us to our first result.

Theorem 1. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ an odd sublinear function and $k \in Q_T$ an odd function, then equation*

$$u'' + g(u) = k(t)$$

has an odd and T -periodic solution. Even more, the set of solutions is bounded.

Proof. Based on the discussion of the first section and the latter remark it is clear that

$$Nu = k - g(u) \in \mathcal{H},$$

and that u is a solution of (1) if and only if u is a fixed point of

$$K(u) = L^{-1}Nu.$$

From Schäfer's theorem it suffices to show that the set

$$\Sigma := \{u \in \mathcal{H} : u = \lambda L^{-1}Nu, \lambda \in (0, 1]\}$$

is bounded. Let us suppose the opposite and let us take $(u_n) \subset \Sigma$ and $(\lambda_n) \subset (0, 1]$ such that

$$u_n = \lambda_n L^{-1}Nu_n$$

and

$$\|u_n\|_\infty \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

The sublinearity of g implies that, for all $\varepsilon > 0$ there exists $M := M(\varepsilon)$ such that $g(t) < M + \varepsilon t$. For $n \in \mathbb{N}$ it follows that

$$\begin{aligned} \|u_n\|_\infty &\leq \lambda_n \|L^{-1}\|_{\mathcal{L}} \|k - g(u_n)\|_\infty \\ &\leq \lambda_n \|L^{-1}\|_{\mathcal{L}} (\|k\|_\infty + M(\varepsilon) + \varepsilon \|u_n\|_\infty) \\ &\leq \lambda_n \varepsilon \|L^{-1}\|_{\mathcal{L}} \|u_n\|_\infty + C(\varepsilon) \end{aligned} \quad (C(\varepsilon) > 0).$$

Consequently,

$$\infty = \lim_{n \rightarrow \infty} \|u_n\|_\infty \leq \lim_{n \rightarrow \infty} \frac{C(\varepsilon)}{1 - \lambda_n \varepsilon \|L^{-1}\|_{\mathcal{L}}},$$

which in any way means that

$$\lim_{n \rightarrow \infty} \lambda_n = \frac{1}{\varepsilon \|L^{-1}\|_{\mathcal{L}}}.$$

By putting $\varepsilon < \|L^{-1}\|^{-1}$ we get

$$\lim_{n \rightarrow \infty} \lambda_n > 1$$

which contradicts the fact that $(\lambda_n) \subset (0, 1]$. □

3. UNIQUENESS OF SOLUTIONS IN \mathcal{H} UNDER CONVEXITY CONDITIONS

Before stating the main result let us remember that we first considered L as an operator with domain in P_T and therefore $\|S\|_{\mathcal{L}} = \|L^{-1}\|_{\mathcal{L}}$ depends only on T .

Theorem 2. *If $g \in C^1(\mathbb{R})$ is an odd function such that $\|g'\|_{\infty} < 2/T^2$ and $k \in Q_T$ is an odd function then there exists a unique solution in \mathcal{H} of equation*

$$u'' + g(u) = k(t).$$

Which between lines tells us that the uniqueness depends on the period T .

Proof. As in Theorem 1, we look for a fixed point of the equation

$$K(u) = L^{-1}Nu \quad (u \in \mathcal{H}).$$

The fact that $\|g'\|_{\infty} < 2/T^2$, implies the existence of $\lambda \in (0, 1)$ such that

$$\|L^{-1}\|_{\mathcal{L}} |g(x) - g(y)| < \lambda |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

Then, if $u, v \in \mathcal{H}$ we obtain

$$\begin{aligned} \|K(u) - K(v)\|_{\infty} &\leq \|L^{-1}\|_{\mathcal{L}} \|Nu - Nv\|_{\infty} \\ &= \|L^{-1}\|_{\mathcal{L}} \|g(v) - g(u)\|_{\infty} \\ &\leq \lambda \|v - u\|_{\infty} = \lambda \|u - v\|_{\infty}. \end{aligned}$$

Banach's fixed point theorem guarantees the existence and uniqueness of a solution of (1) in \mathcal{H} . □

REFERENCES

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